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CONSTRUCTION OF REFINED APPLIED THEORIES FOR A PLATE ON THE BASIS OF THE EQUATIONS OF THE THEORY OF ELASTICITY

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A method is suggested for the construction of an improved theory for a plate with arbitrary smooth contour, which allows to determine the interior state of stress and strain of the plate with an arbitrary degree of accuracy.

The asymptotic analysis of the solutions of elasticity theory problems [1 - 3] has shown that the state of stress of a plate can be separated into an internal state and a boundary layer-type state. The solution of the Kirchhoff theory is the first term of the asymptotics of the internal state of stress of the plate and has an error of order λ (λ is the relative thickness of the plate) for a sufficiently slowly changing external load applied to the edges of the plate.

In [4] boundary value problems are formulated for some homogeneous conditions on the contour of the plate, whose solutions have an error of order λ^2 for the interior state of stress. The constants which occur in the refined boundary conditions are not actually determined. In [5], for a circular plate whose contour is free of stresses, one formulates fundamental equations which assure an error of order λ^3 for the interior state of stress.

1. We consider the elastic equilibrium of a multiply connected plate (Fig. 1). We will assume that the distances between the surfaces Γ_i are sufficiently large in comparison with the thickness, such that the mutual influence of the boundary effects can be neglected. We assume that the edges of the plate are free of stresses and on the surfaces Γ_i we may have boundary conditions of the following type:

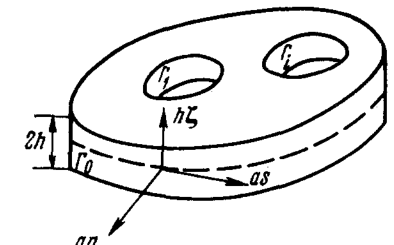

Fig. 1

$$\sigma_n|_\Gamma = N(s,\zeta), \qquad \tau_{ns}|_\Gamma = T(s,\zeta)$$

$$\tau_{nz}|_\Gamma = Z(s,\zeta) \tag{1.1}$$

$$u_n|_\Gamma = u_s|_\Gamma = w|_\Gamma = 0 \tag{1.2}$$

$$\sigma_n|_\Gamma = \tau_{ns}|_\Gamma = w|_\Gamma = 0 \tag{1.3}$$

$$\sigma_n|_\Gamma = u_s|_\Gamma = w|_\Gamma = 0 \tag{1.4}$$

$$u_n|_\Gamma = U(s,\zeta), \qquad \tau_{ns}|_\Gamma = T(s,\zeta), \qquad \tau_{nz}|_\Gamma = Z(s,\zeta) \tag{1.5}$$

Far from the edge the state of stress and strain of the plate is determined by the following expressions:

in the case of bending

$$\sigma_n^\circ = 2\mu\left\{\lambda\zeta R_1(\psi) + \frac{1}{2}\left(\nu + \frac{1}{3}\right)\lambda^3\left(\frac{3}{5}\zeta - \zeta^3\right)\frac{\partial^2\Delta\psi}{\partial n^2}\right\} \tag{1.6}$$

$$\tau_{ns}^{\circ} = 2\mu\left\{(\nu+1)\lambda\zeta R_2(\psi) + \frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3\left(\frac{3}{5}\zeta-\zeta^3\right)R_2(\Delta\psi)\right\}$$

$$\tau_{nz}^{\circ} = 2\mu\nu\lambda^2(1-\zeta^2)\frac{\partial\Delta\psi}{\partial n}$$

$$u_n^{\circ} = a\left\{(\nu+1)\lambda\zeta\frac{\partial\psi}{\partial n} + \frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3\left(\frac{3}{5}\zeta-\zeta^3\right)\frac{\partial\Delta\psi}{\partial n}\right\}$$

$$u_s^{\circ} = a\left\{(\nu+1)\lambda\zeta\frac{1}{H}\frac{\partial\psi}{\partial s} + \frac{1}{2}\left(\nu+\frac{1}{3}\right)\lambda^3\left(\frac{3}{5}\zeta-\zeta^3\right)\frac{1}{H}\frac{\partial\Delta\psi}{\partial s}\right\} \qquad (1.7)$$

$$w^{\circ} = a\left\{-(\nu+1)\psi + \frac{\lambda^2}{2}\left[\frac{17\nu-1}{5}-(\nu-1)\zeta^2\right]\Delta\psi\right\}$$

$$R_1(\psi) = 2\nu\frac{\partial^2\psi}{\partial n^2} + (\nu-1)\left(\frac{1}{H^2}\frac{\partial^2\psi}{\partial s^2} + \frac{a}{RH}\frac{\partial\psi}{\partial n} + n\frac{aR'}{R^2H^3}\frac{\partial\psi}{\partial s}\right)$$

$$R_2(\psi) = \frac{1}{H}\frac{\partial^2\psi}{\partial n\partial s} - \frac{a}{RH^2}\frac{\partial\psi}{\partial s}$$

$$\nu = 1/(1-2\sigma), \qquad \lambda = h/a, \qquad H = 1 + na/R$$

in the case of extension

$$\sigma_n^{\circ} = 2\mu\left\{\left[-l\frac{\partial}{\partial s}S_{12}^*(\varphi,\chi) + m\frac{\partial}{\partial s}S_{11}^*(\varphi,\chi)\right] - \lambda^2\left(\frac{1}{3}-\zeta^2\right)\left[-l\frac{\partial}{\partial s}S_{22}(\varphi) + m\frac{\partial}{\partial s}S_{21}(\varphi)\right]\right\}$$

$$\tau_{ns}^{\circ} = 2\mu\left\{\left[l\frac{\partial}{\partial s}S_{11}^*(\varphi,\chi) + m\frac{\partial}{\partial s}S_{12}^*(\varphi,\chi)\right] - \lambda^2\left(\frac{1}{3}-\zeta^2\right)\left[m\frac{\partial}{\partial s}S_{22}(\varphi) + l\frac{\partial}{\partial s}S_{21}(\varphi)\right]\right\} \qquad (1.8)$$

$$\tau_{nz}^{\circ} = 0$$

$$u_n^{\circ} = a\left\{\left[lS_{11}(\varphi,\chi) + mS_{12}(\varphi,\chi)\right] - \lambda^2\left(\frac{1}{3}-\zeta^2\right)[lS_{21}(\varphi) + mS_{22}(\varphi)]\right\} \qquad (1.9)$$

$$u_s^{\circ} = a\left\{[lS_{12}(\varphi,\chi) - mS_{11}(\varphi,\chi)] - \lambda^2\left(\frac{1}{3}-\zeta^2\right)[lS_{22}(\varphi) - mS_{21}(\varphi)]\right\}$$

$$w^{\circ} = -2a^2\frac{\nu-1}{3\nu-1}\lambda\zeta(\varphi' + \overline{\varphi'})$$

$$S_{11}(\varphi,\chi) = -\frac{1}{2}[-\varkappa(\varphi+\bar\varphi) + (z\overline{\varphi'} + \bar z\varphi') + (\chi+\bar\chi)]$$

$$S_{12}(\varphi,\chi) = \frac{i}{2}[-\varkappa(\varphi-\bar\varphi) + (z\overline{\varphi'} - \bar z\varphi') + (\bar\chi-\chi)]$$

$$S_{21}(\varphi) = a^2\frac{\nu-1}{3\nu-1}(\varphi'' + \overline{\varphi''}), \qquad S_{22}(\varphi) = ia^2\frac{\nu-1}{3\nu-1}(\varphi'' - \overline{\varphi''})$$

$$S_{ij}^*(\varphi,\chi) = S_{ij}(\varphi,\chi)|_{\varkappa=-1}, \quad \varkappa = \frac{3-\sigma}{1+\sigma}, \quad z = x+iy, \quad l = \cos(nx), \quad m = \cos(ny)$$

Here $\psi(n, s)$ is a biharmonic function; $\varphi(z)$, $\chi(z)$ are analytic functions of the complex coordinate z; n, s, ζ are dimensionless curvilinear coordinates (see Fig. 1); μ, σ are elastic constants; $2h$ is the thickness of the plate; a is some characteristic dimension of the plate in the plane.

The purpose of the paper is to formulate in the problem (1.1) – (1.5) boundary conditions for the functions ψ, φ, χ such that the corresponding interior state of stress and strain should have an error $O(\lambda^n)$.

2. We consider the case of the boundary conditions (1.1). According to [2, 3] on the boundary Γ_i we have

$$\sigma_n = \sigma_n^\circ|_{n=0} + 4\mu\lambda^2 \sum_{k=1}^{\infty} l_k(\zeta)\left(\frac{\partial^2 B_k}{\partial s \partial n}\bigg|_{n=0} - \frac{a}{R}\frac{\partial b_k}{\partial s}\right) -$$

$$- \lambda\mu \sum_{p=1}^{\infty}\left\{2\nu F_p'' \frac{c_p}{\lambda^2} - \frac{a}{R}\left[(\nu+1)\frac{F_p''}{\gamma_p^2} - (\nu-1)F_p\right]\frac{\partial C_p}{\partial n}\bigg|_{n=0}\right\} \tag{2.1}$$

$$\tau_{ns} = \tau_{ns}^\circ|_{n=0} + 2\mu\lambda^2 \sum_{k=1}^{\infty} l_k(\zeta)\left(\frac{\partial^2 b_k}{\partial s^2} + \frac{a}{R}\frac{\partial B_k}{\partial n} - \frac{\partial^2 B_k}{\partial n^2}\right)\bigg|_{n=0} -$$

$$- \lambda\mu \sum_{p=1}^{\infty}\left[(\nu+1)\frac{F_p''}{\gamma_p^2} - (\nu-1)F_p\right]\left(\frac{\partial^2 C_p}{\partial n \partial s}\bigg|_{n=0} - \frac{a}{R}\frac{\partial c_p}{\partial s}\right) \tag{2.2}$$

$$\tau_{nz} = \tau_{nz}^\circ|_{n=0} + 2\mu\lambda \sum_{k=1}^{\infty} l_k'(\zeta)\frac{\partial b_k}{\partial s} + 2\mu\nu \sum_{p=1}^{\infty} F_p' \frac{\partial C_p}{\partial n}\bigg|_{n=0} \tag{2.3}$$

$$u_n = u_n^\circ|_{n=0} + 2\lambda^2 a \sum_{k=1}^{\infty} l_k(\zeta)\frac{\partial b_k}{\partial s} - \frac{\lambda a}{2}\sum_{p=1}^{\infty}\left[(\nu+1)\frac{F_p''}{\gamma_p^2} - (\nu-1)F_p\right]\frac{\partial C_p}{\partial n}\bigg|_{n=0} \tag{2.4}$$

$$u_s = u_s^\circ|_{n=0} - 2\lambda^2 a \sum_{k=1}^{\infty} l_k(\zeta)\frac{\partial B_k}{\partial n}\bigg|_{n=0} - \frac{\lambda a}{2}\sum_{p=1}^{\infty}\left[(\nu+1)\frac{F_p''}{\gamma_p^2} - (\nu-1)F_p\right]\frac{\partial c_p}{\partial s} \tag{2.5}$$

$$w = w^\circ|_{n=0} + \frac{a}{2}\sum_{p=1}^{\infty}\left[(\nu+1)\frac{F_p'''}{\gamma_p^2} + (3\nu+1)F_p'\right]c_p \tag{2.6}$$

Here $B_k(n, s)$, $C_p(n, s)$ determine the boundary layer solution of the equations

$$\Delta B_k - \frac{\sigma_k^2}{\lambda^2}B_k = 0, \qquad \Delta C_p - \frac{\gamma_p^2}{\lambda^2}C_p = 0 \tag{2.7}$$

in the case of bending

$$F_p(\zeta) = \left(\sin\gamma_p - \frac{\cos\gamma_p}{\gamma_p}\right)\sin\gamma_p\zeta + \zeta\cos\gamma_p\cos\gamma_p\zeta, \qquad l_k(\zeta) = \frac{\sin\sigma_k\zeta}{\sigma_k}$$

$$\sin 2\gamma_p - 2\gamma_p = 0, \qquad \cos\sigma_k = 0$$

in the case of extension

$$F_p(\zeta) = \left(\cos\gamma_p + \frac{\sin\gamma_p}{\gamma_p}\right)\cos\gamma_p\zeta + \zeta\sin\gamma_p\sin\gamma_p\zeta, \quad l_k(\zeta) = \cos\sigma_k\zeta$$

$$\sin 2\gamma_p + 2\gamma_p = 0, \qquad \sin\sigma_k = 0$$

Here $F_p(\zeta)$ are the Papkovich functions. The following generalized orthogonality condition holds for them:

$$\int_{-1}^{1}(F_p''F_t'' - \gamma_p^2\gamma_t^2 F_p F_t)\,d\zeta = \begin{cases} 4\gamma_t^4, & p = t \\ 0, & p \neq t \end{cases} \tag{2.8}$$

In [2, 3] the asymptotic representations of the solutions of Eqs. (2.7) are given in terms of their boundary values $b_k(s)$, $c_p(s)$. In the sequel we will assume that these functions can be represented in the form

$$b_k(s) = \sum_{i=0}^{\infty} \lambda^i b_{ki}(s), \qquad c_p(s) = \sum_{i=0}^{\infty} \lambda^i c_{pi}(s) \tag{2.9}$$

The coefficients b_{ki}, c_{pi} of these expansions are determined from infinite recursive systems [2, 3].

First we consider the case of bending. We introduce the notation

$$M_n = \frac{1}{\lambda}\int_{-1}^{1} N\zeta^n d\zeta, \quad G_n = \frac{1}{\lambda}\int_{-1}^{1} T\zeta^n d\zeta \quad (n = 1, 3, \quad Q_0 = \frac{1}{\lambda^2}\int_{-1}^{1} Z d\zeta$$

$$Q_2 = \frac{1}{\lambda}\int_{-1}^{1} Z\zeta^2 d\zeta, \quad T_k = \frac{1}{\lambda}\int_{-1}^{1} Tl_k(\zeta)\, d\zeta, \quad N_k^{(1)} = \frac{1}{\lambda}\int_{-1}^{1} Nl_k(\zeta)\, d\zeta \quad (k = 1, 2, \ldots) \tag{2.10}$$

$$N_p^{(2)} = \frac{1}{2\lambda}\int_{-1}^{1} N\left[(\nu - 1) F_p - \frac{\nu + 1}{\gamma_p^2} F_p''\right] d\zeta,$$

$$Z_p = \frac{1}{2\lambda}\int_{-1}^{1} Z\left[(3\nu + 1) F_p' + \frac{\nu + 1}{\gamma_p^2} F_p''\right] d\zeta \qquad (p = 1, 2, \ldots)$$

According to [2], we have

$$b_{k0} = 0, \qquad b_{k1} = \frac{2(\nu + 1)(-1)^{k+1}}{\sigma_k^3} R_2(\psi) - \frac{1}{2\mu} T_k$$

$$b_{k2} = \frac{2(-1)^{k+1}}{\sigma_k^4}\left[\frac{\partial}{\partial s} R_1(\psi) - (\nu + 1)\frac{a}{2R} R_2(\psi)\right]_{n=0} + \frac{5a}{2R}\frac{b_{k1}}{\sigma_k} - \tag{2.11}$$

$$- \frac{2(-1)^{k+1}}{\sigma_k}\sum_{p=1}^{\infty}\frac{\cos^2\gamma_p}{\gamma_p^2 - \sigma_k^2}\gamma_p\left(1 - \nu\frac{\gamma_p + \sigma_k}{\gamma_p - \sigma_k}\right)c_{p2}' - \frac{1}{2\mu\sigma_k}\left(\frac{\partial N_k^{(1)}}{\partial s} - \frac{a}{2R} T_k\right), \ldots$$

$$c_{p0} = c_{p1} = 0, \qquad c_{p2} = \alpha_p R_1(\psi)\,|_{n=0} + \frac{1}{2\mu}\beta_p, \ldots \tag{2.12}$$

$$\alpha = -2(\nu - 1) M \mathbf{q}_1, \qquad \beta = -M\mathbf{r}$$

Here M is the inverse matrix of the infinite system of linear algebraic equations which determines c_{pi} and it is independent of the load and of the geometry of the plate; α, β, $\mathbf{q}_1$, $\mathbf{r}$ are infinite dimensional vectors with components α_p, β_p, $\gamma_p^{-1}\sin^2\gamma$, $Z_p + \gamma_p N_p^{(1)}$ $(p = 1, 2, \ldots)$, respectively.

On the basis of the formulas (2.1) – (2.3) we construct the expressions for the bending moment and for the generalized shearing force on Γ_i Taking into account (2.10), we have

$$M_1 = \frac{4}{3}\mu R_1(\psi)\Big|_{n=0} + \lambda 2\mu\left\{4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^2} b_{k1}' - \frac{\nu - 1}{2}\frac{a}{R}\sum_{p=1}^{\infty}\gamma_p F_p^* c_{p2}\right\} +$$

$$+ \lambda^2 2\mu\left\{-\frac{1}{5}\left(\nu + \frac{1}{3}\right)\frac{\partial^2\Delta\psi}{\partial n^2}\Big|_{n=0} +\right.$$

$$+ 4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^3}\left[\sigma_k b_{k2} - \left(\frac{a}{R}\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{a}{2R}\right) b_{k1}\right] - \frac{\nu - 1}{2}\frac{a}{R}\sum_{p=1}^{\infty}\gamma_p F_p^* c_{p3} +$$

$$\left. + \frac{\nu - 1}{2}\left(\frac{a^2}{2R^2} - \frac{\partial^2}{\partial s^2}\right)\sum_{p=1}^{\infty} F_p^* c_{p2}\right\} + O(\lambda^3) \tag{2.13}$$

$$Q_0 + \frac{\partial G_1}{\partial s} = \frac{4}{3}\mu\left[2\nu\frac{\partial\Delta\psi}{\partial n} + (\nu+1)\frac{\partial}{\partial s}R_2(\psi)\right]_{n=0} +$$

$$+\lambda 2\mu\left\{4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^2}\frac{\partial}{\partial s}\frac{a}{R}b_{k1} + \frac{\nu-1}{2}\frac{\partial^2}{\partial s^2}\sum_{p=1}^{\infty}\gamma_p F_p{}^* c_{p2}\right\} +$$

$$+\lambda^2 2\mu\left\{-\frac{1}{5}\left(\nu+\frac{1}{3}\right)\frac{\partial}{\partial s}R_2(\Delta\psi)\Big|_{n=0} + 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^3}\left[2\sigma_k\frac{\partial}{\partial s}\frac{a}{R}b_{k2} +\right.\right.$$

$$\left.+\frac{\partial}{\partial s}\left(2\frac{\partial^2}{\partial s^2} - \frac{a^2}{R^2}\right)b_{k1}\right] + \frac{\nu-1}{2}\frac{\partial^2}{\partial s^2}\sum_{p=1}^{\infty}\gamma_p F_p{}^* c_{p3} -$$

$$\left.-\frac{\nu-1}{2}\frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\frac{a}{2R} + \frac{a}{R}\frac{\partial}{\partial s}\right)\sum_{p=1}^{\infty}F_p{}^* c_{p2}\right\} + O(\lambda^3) \qquad (2.14)$$

$$F_p{}^* = \int_{-1}^{1}\zeta F_p(\zeta)\,d\zeta = -4\,\frac{\sin^2\gamma_p}{\gamma_p^2}$$

Then from (2.3) we obtain

$$Q_2 = \lambda\frac{8\mu\nu}{15}\frac{\partial\Delta\psi}{\partial n}\Big|_{n=0} + 4\mu\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sigma_k^2-2}{\sigma_k^3}(\lambda b_{k1}' + \lambda^2 b_{k2}' + \ldots) -$$

$$-4\mu\nu\sum_{p=1}^{\infty}F_p{}^*\left[\gamma_p c_{p2} + \lambda\left(\gamma_p c_{p3} - \frac{a}{2R}c_{p2}\right) + \ldots\right] \qquad (2.15)$$

From (2.15) we have

$$\sum_{p=1}^{\infty}\gamma_p F_p{}^* c_{p2} = -\frac{Q_2}{4\mu\nu}$$

$$\sum_{p=1}^{\infty}\gamma_p F_p{}^* c_{p3} = \frac{2}{15}\frac{\partial\Delta\psi}{\partial n}\Big|_{n=0} + \frac{1}{\nu}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sigma_k^2-2}{\sigma_k^3}b_{k1}' +$$

$$+\frac{a}{2R}\sum_{p=1}^{\infty}F_p{}^* c_{p2}, \ldots \qquad (2.16)$$

Substituting (2.11), (2.16) into (2.13), (2.14) and taking into account (2.12) we obtain boundary conditions for the determination of $\psi(n, s)$ which is correct within quantities of order λ^2 compared to unity.

$$R_1(\psi)|_{n=0} + \lambda 12A(\nu+1)\frac{\partial}{\partial s}R_2(\psi)|_{n=0} + \lambda^2\left[-\frac{3}{10}\left(\nu+\frac{1}{3}\right)\frac{\partial^2\Delta\psi}{\partial n^2} +\right.$$

$$\left.+\frac{6}{5}(\nu+1)\left(\frac{\partial}{\partial s}\frac{a}{R} - \frac{2}{3}\frac{a}{R}\frac{\partial}{\partial s}\right)R_2(\psi)\right]_{n=0} + O(\lambda^3) =$$

$$= \frac{1}{2\mu}\left\{\frac{3}{2}M_1 + \lambda\left[6\frac{d}{ds}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^2}T_k - \frac{3(\nu-1)}{8\nu}\frac{a}{R}Q_2\right] +\right.$$

$$+\lambda^2\left[\frac{1}{4}\left(\frac{3\nu+1}{2\nu}\frac{a}{R}\frac{d}{\partial s} - 3\frac{d}{ds}\frac{a}{R}\right)(G_3 - 3G_1) - \frac{3(\nu-1)}{10\nu}\frac{a}{R}\frac{dG_1}{ds} -\right.$$

$$\left.-\frac{1}{2}\frac{d^2}{ds^2}\left(M_3 - \frac{3}{5}M_1\right) + \frac{3}{40}\frac{\nu-1}{\nu}\frac{a}{R}Q_0 + \frac{d^2}{ds^2}L\right] + O(\lambda^3) \qquad (2.17)$$

$$\left[2\nu\frac{\partial\Delta\psi}{\partial n}+(\nu+1)\frac{\partial}{\partial s}R_2(\psi)\right]_{n=0}+\lambda 12A(\nu+1)\frac{\partial}{\partial s}\frac{a}{R}R_2(\psi)\Big|_{n=0}+$$
$$+\lambda^2\left[-\frac{3}{10}\left(\nu+\frac{1}{3}\right)\frac{\partial}{\partial s}R_2(\Delta\psi)+\frac{6}{5}(\nu+1)\frac{\partial}{\partial s}\left(\frac{a^2}{R^2}+\frac{2}{3}\frac{\partial^2}{\partial s^2}\right)R_2(\psi)\right]_{n=0}+$$
$$+O(\lambda^3)=\frac{1}{2\mu}\left\{\frac{3}{2}\left(Q_0+\frac{dG_1}{ds}\right)+\lambda\left[6\frac{d}{ds}\frac{a}{R}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^2}T_k+\right.\right.$$
$$\left.+\frac{3(\nu-1)}{8\nu}\frac{d^2}{ds^2}Q_2\right]+\lambda^2\left[-\frac{1}{4}\left(\frac{3\nu+1}{2\nu}\frac{d^2}{ds^2}+3\frac{a^2}{R^2}\right)(G_3-3G_1)+\right.$$
$$+\frac{3(\nu-1)}{10\nu}\frac{d^3G_1}{ds^3}-\frac{1}{2}\frac{d}{ds}\frac{a}{R}\frac{d}{ds}\left(M_3-\frac{3}{5}M_1\right)-$$
$$\left.-\frac{3(\nu-1)}{40\nu}\frac{d^2Q_0}{ds^2}+\frac{d}{ds}\frac{a}{R}\frac{d}{ds}L\right]+O(\lambda^3)\qquad(2.18)$$

$$L=3(1-\nu)\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p^2}\left(\beta_p+\frac{3}{2}M_1\alpha_p\right)+$$
$$+12\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{\gamma_p\cos^2\gamma_p}{\sigma_k^3(\gamma_p^2-\sigma_k^2)}\left(1-\nu\frac{\gamma_p+\sigma_k}{\gamma_p-\sigma_k}\right)\left(\beta_p+\frac{3}{2}M_1\alpha_p\right)$$
$$A=\sum_{k=1}^{\infty}\frac{1}{\sigma_k^5}\approx\begin{cases}0.1050, & \sigma_k>0\\ -0.1050, & \sigma_k<0\end{cases}$$

Obviously, the process of the construction of the boundary conditions for ψ can be continued as much as desired in the case of sufficient smoothness of both the contour and the load. In the first approximation the relations (2.17), (2.18) coincide with the boundary conditions of Kirchhoff's theory. From (2.17), (2.18) we can also see that in the general case of given stresses under bending, the boundary conditions for ψ can be constructed without the inversion of the infinite system, with an accuracy to quantities of order λ compared with unity.

In the case when the surface Γ_i is free of stresses, the boundary conditions for ψ can be constructed without the inversion of the infinite system with an accuracy to quantities of order λ^2 compared with unity. In this case $c_{p2}=0$ [6] and from (2.17), (2.18) we obtain (2.19)

$$R_1(\psi)|_{n=0}+\lambda 12A(\nu+1)\frac{\partial}{\partial s}R_2(\psi)\Big|_{h=0}+\lambda^2\left[-\frac{3}{10}\left(\nu+\frac{1}{3}\right)\frac{\partial^2\Delta\psi}{\partial n^2}+\right.$$
$$\left.+\frac{6(\nu+1)}{5}\left(\frac{\partial}{\partial s}\frac{a}{R}-\frac{2}{3}\frac{a}{R}\frac{\partial}{\partial s}\right)R_2(\psi)\right]_{n=0}+O(\lambda^3)=0$$

$$\left[2\nu\frac{\partial\Delta\psi}{\partial n}+(\nu+1)\frac{\partial}{\partial s}R_2(\psi)\right]_{n=0}+\lambda 12A(\nu+1)\frac{\partial}{\partial s}\frac{a}{R}R_2(\psi)\Big|_{n=0}+$$
$$+\lambda^2\left[-\frac{3}{10}\left(\nu+\frac{1}{3}\right)\frac{\partial}{\partial s}R_2(\Delta\psi)+\frac{6}{5}(\nu+1)\frac{\partial}{\partial s}\left(\frac{a^2}{R^2}+\frac{2}{3}\frac{\partial^2}{\partial s^2}\right)R_2(\psi)\right]_{n=0}+$$
$$+O(\lambda^3)=0$$

We consider now the case of the extension-compression of the plate. In the same way as before one can construct a process in order to obtain boundary conditions for the determination of the analytic functions φ and χ with an arbitrary degree of accuracy. In the given case, without the inversion of the infinite system, one can formulate three approximations of the boundary conditions for the plate acted upon by an arbitrary load (2.20) and five approximations for the free contour (2.21)

$$-2\mu\frac{\partial}{\partial s}[S_{12}^*(\varphi,\chi)-iS_{11}^*(\varphi,\chi)]_{n=0}+O(\lambda^3)=X_{n0}+iY_{n0}+$$
$$+\lambda\frac{\nu-1}{2\nu}\left[\left(-l\frac{a}{R}-m\frac{d}{ds}\right)+i\left(-m\frac{a}{R}+l\frac{d}{ds}\right)\right]Z_1+$$
$$+\lambda^2\frac{\nu-1}{4\nu}\left\{\left[\left(-l\frac{a}{R}-m\frac{d}{ds}\right)+i\left(-m\frac{a}{R}+l\frac{d}{ds}\right)\right]\frac{d}{ds}\left(\frac{1}{3}T_0-T_2\right)+\right.$$
$$\left.+\left[\left(l\frac{d}{ds}-m\frac{a}{R}\right)+i\left(m\frac{d}{ds}+l\frac{a}{R}\right)\right]\frac{d}{ds}\left(\frac{1}{3}N_0-N_2\right)\right\}+O(\lambda^3)\quad(2.20)$$

$$-\frac{\partial}{\partial s}[S_{12}^*(\varphi,\chi)-iS_{11}^*(\varphi,\chi)]_{n=0}+\lambda^4\frac{1-\nu}{45\nu}\left\{(l+im)\left[\frac{\partial^2}{\partial s^2}S_3(\varphi)+\right.\right.$$
$$\left.\left.+\frac{a}{R}\frac{\partial}{\partial s}S_4(\varphi)\right]_{n=0}+(m-il)\left[\frac{\partial^2}{\partial s^2}S_4(\varphi)-\frac{a}{R}\frac{\partial}{\partial s}S_3(\varphi)\right]_{n=0}\right\}+O(\lambda^5)=0$$
$$(2.21)$$

$$S_3(\varphi)=-l\frac{\partial}{\partial s}S_{22}(\varphi)+m\frac{\partial}{\partial s}S_{21}(\varphi),\qquad S_4(\varphi)=l\frac{\partial}{\partial s}S_{21}(\varphi)+m\frac{\partial}{\partial s}S_{22}(\varphi)$$

Here we have denoted

$$X_{n0}=\frac{1}{2}\int_{-1}^{1}(Nl-Tm)\,d\zeta,\quad Y_{n0}=\frac{1}{2}\int_{-1}^{1}(Nm+Tl)\,d\zeta,\qquad Z_1=\frac{1}{2}\int_{-1}^{1}Z\zeta\,d\zeta$$
$$N_i=\frac{1}{2}\int_{-1}^{1}N\zeta^i\,d\zeta,\qquad T_i=\frac{1}{2}\int_{-1}^{1}T\zeta^i\,d\zeta\qquad(i=0,2)$$

3. We consider the case when the contour Γ_i of the plate is rigidly fixed. In the bending problem, satisfying the boundary conditions (1.2), we obtain

$$c_{p0}=c_{p1}=0,\qquad c_{p2}=\alpha_p R_1(\psi)|_{n=0},\ldots\qquad(3.1)$$

Substituting (3.1) into the expressions (2.4), (2.6) and taking into account (1.2), (1.9) we obtain with an accuracy to quantities of order λ compared with unity, the boundary conditions for the determination of $\psi\ (n,\ s)$

$$\frac{\partial\psi}{\partial n}\Big|_{n=0}+\lambda\left[-\frac{6\nu(\nu-1)}{\nu+1}\Delta\psi\Big|_{n=0}\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}\alpha_p\right]+O(\lambda^2)=0$$
$$\psi|_{n=0}+O(\lambda^2)=0\qquad(3.2)$$

As before, this process can be continued as much as desired.

In the case of extension one can construct in the same way boundary conditions for the determination of φ, χ

$$(m+il)[S_{12}(\varphi,\chi)-iS_{11}(\varphi,\chi)]_{n=0}+\lambda a(\varphi'+\overline{\varphi'})\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p}\delta_p+O(\lambda^2)=0$$
$$\delta=8\frac{\nu(\nu-1)^2}{3\nu-1}M_*q_2\qquad(3.3)$$

where δ, q_2 are infinite dimensional vectors with components δ_p, $\gamma_p^{-1}\sin^2\gamma_p$ $(p=1,2,\ldots)$ respectively, M_* is the inverse matrix of the infinite system given in [3].

4. We shall consider the surface Γ_i articulated, if on it the conditions (1.3) or (1.4) hold. In these cases the matrices of the systems of equations for the determination of b_{ki} and c_{pi} turn out to be in the diagonal form [7], and the boundary conditions for the determination of the penetrating part of the state of stress can be obtained in explicit form with any degree of accuracy. We give two terms of the expansions of these boundary conditions for the cases (1.3) and (1.4), namely the relations (4.1), (4.3) and (4.2), (4.4) respectively.

In the bending problem

$$R_1(\psi)|_{n=0} + \lambda\left[12A(\nu+1)\frac{\partial}{\partial s}R_2(\psi) - \frac{6(\nu-1)^2}{\nu+1}\frac{a}{R}B_1\Delta\psi\right]_{n=0} + O(\lambda^2) = 0 \qquad (4.1)$$

$$\psi|_{n=0} + O(\lambda^2) = 0, \qquad B_1 = \sum_{p=1}^{\infty}\frac{\sin^4\gamma_p}{\gamma_p^5} \approx \begin{cases} -0.01476, & \operatorname{Re}\gamma_p > 0 \\ 0.01476, & \operatorname{Re}\gamma_p < 0 \end{cases}$$

$$R_1(\psi)|_{n=0} + \lambda\left[-\frac{6(\nu-1)^2}{\nu+1}\frac{a}{R}B_1\Delta\psi\right]_{n=0} + O(\lambda^2) = 0, \qquad \psi|_{n=0} + O(\lambda^2) = 0 \qquad (4.2)$$

in the extension problem

$$(l - im)\frac{\partial}{\partial s}[S_{12}^*(\varphi,\chi) - iS_{11}^*(\varphi,\chi)]_{n=0} +$$

$$+\lambda\left[\frac{(\nu-1)^2 4B_2}{(\nu+1)(3\nu-1)}\left(\frac{a}{R} - i\frac{\partial}{\partial s}\right)(\varphi' + \overline{\varphi'})\right]_{n=0} + O(\lambda^2) = 0 \qquad (4.3)$$

$$\left[-l\frac{\partial}{\partial s}S_{12}^*(\varphi,\chi) + m\frac{\partial}{\partial s}S_{11}^*(\varphi,\chi)\right]_{n=0} +$$

$$+\lambda\left[-\frac{4(\nu-1)^2 B_2}{(\nu+1)(3\nu-1)}\frac{a}{R}(\varphi' + \overline{\varphi'})\right]_{n=0} + O(\lambda^2) = 0$$

$$[lS_{12}(\varphi,\chi) - mS_{11}(\varphi,\chi)]_{n=0} + O(\lambda^2) = 0 \qquad (4.4)$$

$$B_2 = \sum_{p=1}^{\infty}\frac{\sin^4\gamma_p}{\gamma_p^5} \approx \begin{cases} -0.002999, & \operatorname{Re}\gamma_p > 0 \\ 0.002999, & \operatorname{Re}\gamma_p < 0 \end{cases}$$

5. The mixed boundary conditions of type (1.5) present interest. They are obtained, in particular, in the case of shrink fitting in the presence of friction forces. As in the previous case, one can construct here the boundary conditions for ψ, φ, χ in explicit form with any degree of accuracy. We give two terms of the expansions for the cases of bending and extension.

In the bending problem

$$\frac{\partial\psi}{\partial n}\bigg|_{n=0} + O(\lambda^2) = \frac{3}{2a(\nu+1)}\left[U_1 + \lambda\frac{\nu-1}{8\nu}\frac{a}{\mu}Q_2 + O(\lambda^2)\right] \qquad (5.1)$$

$$\left[2\nu\frac{\partial\Delta\psi}{\partial n} + (\nu+1)\frac{\partial}{\partial s}R_2(\psi)\right]_{n=0} + \lambda 12A(\nu+1)\frac{\partial}{\partial s}\frac{a^2}{R^2}\frac{\partial\psi}{\partial s}\bigg|_{n=0} + O(\lambda^2) =$$

$$= \frac{3}{4\mu}\left\{Q_0 + \frac{dG_1}{ds} + \lambda\left[-8\frac{\mu}{a}\frac{d}{ds}\frac{a}{R}\frac{d}{ds}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^2}U_k^{(1)} + \right.\right.$$

$$+4\frac{d}{ds}\frac{a}{R}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sigma_k^2}T_k - \frac{\mu}{a}\frac{2(\nu-1)}{\nu+1}\frac{d}{ds}\frac{a}{R}\frac{d}{ds}\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p^5}U_p^{(2)} +$$

$$\left.\left. + \frac{\nu-1}{4\nu}\frac{d^2Q_2}{ds^2}\right] + O(\lambda^2)\right\}$$

$$U_1 = \frac{1}{\lambda} \int_{-1}^{1} U\zeta d\zeta, \qquad U_k^{(1)} = \frac{1}{\lambda} \int_{-1}^{1} U l_k(\zeta)\, d\zeta, \qquad U_p^{(2)} = \frac{1}{\lambda} \int_{-1}^{1} U F_p'' d\zeta \quad (5.2)$$

in the extension problem

$$a\,[lS_{11}(\varphi,\chi) + mS_{12}(\varphi,\chi)]_{n=0} + O(\lambda^2) = U_0 + \lambda \frac{a}{\mu}\frac{\nu-1}{4\nu} Z_1 + O(\lambda^2)$$

$$2\mu\left[l\frac{\partial}{\partial s} S_{11}^*(\varphi,\chi) + m\frac{\partial}{\partial s} S_{12}^*(\varphi,\chi)\right]_{n=0} + O(\lambda^2) = T_0 +$$

$$+ \lambda \frac{\nu-1}{2\nu}\left[\frac{d}{ds} Z_1 - \frac{4\nu}{\nu+1}\frac{\mu}{R}\frac{d}{ds}\sum_{p=1}^{\infty}\frac{\sin^2\gamma_p}{\gamma_p^5} U_p^{(3)}\right] + O(\lambda^2) \qquad (5.3)$$

$$U_0 = \frac{1}{2}\int_{-1}^{1} U d\zeta, \qquad U_p^{(3)} = \frac{1}{2}\int_{-1}^{1} U F_p'' d\zeta$$

6. In conclusion we prove that for the boundary conditions of the type (1.1) and (1.5) in the case of axisymmetrical deformation, the penetrating solution can be determined exactly without making use of the asymptotic method.

Indeed, considering the boundary conditions (1.1) and taking into account that no stress function depends on s, we have in the case of bending

$$\zeta N(\zeta) = 2\mu\left[\lambda\zeta^2 R_1(\psi) + \frac{\lambda^3}{2}\left(\nu + \frac{1}{3}\right)\left(\frac{3}{5}\zeta^2 - \zeta^4\right)\frac{d^2\Delta\psi}{dn^2}\right]_{n=0} -$$

$$- \frac{2\mu\nu\zeta}{\lambda}\sum_{p=1}^{\infty} F_p'' c_p + \lambda\mu\zeta\sum_{p=1}^{\infty}\left[(1-\nu)F_p + (\nu+1)\frac{F_p''}{\gamma_p^2}\right]\frac{dC_p}{dn}\Big|_{n=0} \qquad (6.1)$$

$$Z(\zeta) = 2\mu\nu\lambda^2(1-\zeta^2)\frac{d\Delta\psi}{dn}\Big|_{n=0} + 2\mu\nu\sum_{p=1}^{\infty} F_p'\frac{dC_p}{dn}\Big|_{n=0} \qquad (6.2)$$

From (6.2), integrating from zero to ζ, we obtain

$$\sum_{p=1}^{\infty} F_p \frac{dC_p}{dn}\Big|_{n=0} = \frac{1}{2\mu\nu}\int_0^{\zeta} Z d\zeta - \lambda^2\left(\zeta - \frac{\zeta^3}{3}\right)\frac{d\Delta\psi}{dn}\Big|_{n=0} \qquad (6.3)$$

Substituting (6.3) into (6.1), (6.2) and integrating from -1 to 1, we obtain the following boundary conditions for ψ:

$$R_1(\psi)|_{n=0} = \frac{3}{4\mu\lambda}\left[M_1 - \lambda\frac{\nu-1}{\nu}\left(\frac{Q_2}{4} + \frac{Q_0}{5}\right)\right], \qquad \frac{d\Delta\psi}{dn}\Big|_{n=0} = \frac{3}{8\mu\nu}\frac{Q_0}{\lambda^2} \quad (6.4)$$

In the case of extension we obtain in the same way

$$\frac{\partial}{\partial s}[-S_{12}^*(\varphi,\chi) + iS_{11}^*(\varphi,\chi)]_{n=0} = \frac{1}{2\mu}\left[X_{n0} + iY_{n0} + \lambda\frac{1-\nu}{2\nu}\frac{z}{a}\Big|_{n=0} Z_1\right] \quad (6.5)$$

Formula (6.5) does not take into account axisymmetric torsion, for which the boundary conditions have the form

$$\left[l\frac{\partial}{\partial s} S_{11}^*(\varphi,\chi) + m\frac{\partial}{\partial s} S_{12}^*(\varphi,\chi)\right]_{n=0} = \frac{T_0}{2\mu} \qquad (6.6)$$

In the case when condition (1.5) is given on the contour, we have:

for bending

$$\left.\frac{d\psi}{dn}\right|_{n=0} = \frac{1}{\lambda}\frac{3}{2(\nu+1)}\left[\frac{U_1}{a} + \frac{\lambda}{4\mu\nu}\left(\frac{\nu-1}{2}Q_2 + \frac{\nu+1}{5}Q_0\right)\right]$$

$$\left.\frac{d\Delta\psi}{dn}\right|_{n=0} = \frac{3}{8\mu\nu}\frac{Q_0}{\lambda^2} \qquad (6.7)$$

for extension

$$[lS_{11}(\rho,\chi) + mS_{12}(\rho,\chi)]_{n=0} = U_0 + \frac{a}{\mu}\frac{\nu-1}{4\nu}\lambda Z_1 \qquad (6.8)$$

In the given formulas $\operatorname{Re}\gamma_p > 0$, $\sigma_k > 0$ for Γ_0 and $\operatorname{Re}\gamma_p < 0$ $\sigma_k < 0$ for Γ_i $(i = 1, 2, \ldots, n)$.

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